Ramsey Theory and the Hales-Jewett Theorem

Slides by Peter Sinclair
Introduction

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Two less trivial results are van der Waerden’s Theorem (1927) and the Hales-Jewett Theorem (1963). A proof of each of these theorems will be presented in this talk.
Definitions

An arithmetic progression is a sequence of the form

\[ a, a + d, a + 2d, a + 3d, \ldots \]

for some integers \( a \) and \( d \). A \( k \)-term arithmetic progression is a finite arithmetic progression with \( k \) terms, ie

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An \( r \)-colouring of a set \( A \) is a function \( c : A \to B \), where \( |B| = r \). Usually we use \( B = \{1, \ldots, r\} \), although any set of \( r \) elements can be used. We say that two elements \( x, y \in A \) have the same colour if \( c(x) = c(y) \), and that a set \( S \subset A \) is monochromatic if every element in \( S \) has the same colour.
Van der Waerden’s Theorem

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We label the least $n$ with this property $W(k, r)$. 
Let $A = \{1, \ldots, k\}$ and let $A' = A \cup \{\ast\}$ for some $\ast \not\in A$. For all $n \in \mathbb{N}$, we will call elements of $A^n$ words and elements of $(A')^n$ roots.
More Definitions

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Given a root \( \tau \in (A')^n \) and a symbol \( a \in A \) we write \( \tau(a) \) for the word created by replacing each instance of \( \ast \) in \( \tau \) with \( a \). For example, if \( \tau = *1*3* \in (A')^5 \) then \( \tau(2) = 21232 \in A^5 \).
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A combinatorial line on a root $\tau$ is the set of words

$$L_{\tau} = \{\tau(1), \tau(2), \tau(3), \ldots, \tau(k)\}$$

For example, if $A = \{1, 2, 3, 4\}$ and $\tau = \ast 1 \ast 3 \ast \in (A')^5$ then

$$L_{\tau} = \{11131, 21232, 31333, 41434\}$$
The Hales-Jewett Theorem

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Let $A = \{1, \ldots, k\}$ and fix $r \geq 1$. There exists a dimension $n$ such that if $A^n$ is partitioned into $r$ subsets, one of the subsets must contain a complete combinatorial line.

We label the least $n$ with this property $H(k, r)$. 
We will prove the Hales-Jewett Theorem by induction on $k$. Trivially, $H(1, r) = 1$ for all $r$ and $H(k, 1) = 1$ for all $k$.

Consider the case $H(2, r)$. Take $n = r$, and notice that two of the points $(1, \ldots, 1, 1), (1, \ldots, 1, 2), \ldots, (2, \ldots, 2, 2)$ must have the same colour since there are $n + 1 = r + 1 > r$ such points. Since any two of these points form a combinatorial line, $H(2, r) \leq r$. In fact, it can be shown that $H(2, r) = r$.

We have just shown that HJT holds whenever $k = 1$, $k = 2$, or $r = 1$. Fix $r > 2$ and $k > 1$, and suppose that the Hales-Jewett Theorem also holds for $r$ and $k - 1$. 
Proof of the Hales-Jewett Theorem

Let $t$ be a large parameter to be chosen and let $N_1, \ldots, N_t$ be a rapidly growing sequence. Set $S_i = N_1 + \ldots + N_i$ and $N = S_t = N_1 + \ldots + N_t$. Let $c$ be an $r$-colouring of $A^N$. 
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We will first define the value of each $N_i$ in decreasing order, and then use these definitions to determine a value of $t$ which makes $N \geq H(k, r)$.
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Write $A^N$ as $A^{S_t-1} \times A^{N_t}$. For each $x \in A^{N_t}$, define an $r$-colouring $c_x$ of $A^{S_t-1}$ by defining $c_x(y) = c(y, x)$ for all $y \in A^{S_t-1}$. Note that there are at most $r^{k^{S_t-1}}$ such colourings.
Proof of the Hales-Jewett Theorem

Now define a new colouring $d : A^{N_t} \to \{ c_x : x \in A^{N_t} \}$ by setting $c_x$ as the colour of $x$ for each $x \in A^{N_t}$. 
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Consider $\{1, 2\}^{N_t} \subset A^{N_t}$. If $N_t \geq H(2, r^{k^{S_t-1}}) = r^{k^{S_t-1}}$, we can find a monochromatic combinatorial line in $\{1, 2\}^{N_t}$; let $z_t \in \{1, 2, *\}^{N_t}$ be the root of this combinatorial line.
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Note that $z_t$ also induces a combinatorial line in $A^{N_t}$, although only $z_t(1)$ and $z_t(2)$ necessarily share a colour. In fact, we have

$$c(y, z_t(1)) = c_{z_t(1)}(y) = c_{z_t(2)}(y) = c(y, z_t(2))$$

for all $y \in A^{S_{t-1}}$. 
Proof of the Hales-Jewett Theorem

Repeat the process for $N_{t-1}$. For each $x \in A^{N_{t-1}}$, define an $r$-colouring $c_x$ of $A^{S_{t-2}} \times A$ by defining $c_x(y, a) = c(y, x, z_t(a))$ for all $y \in A^{S_{t-2}}$ and $a \in A$. This time, there are at most $r(k-1)k^{S_{t-2}}$ such colourings, since $a = 1$ and $a = 2$ induce the same colouring for all $x, y$. 
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Again, colour $A^{N_{t-1}}$ by choosing $c_x$ as the colour of $x$ for each $x \in A^{N_{t-1}}$. If $N_{t-1} \geq r^{(k-1)k^{S_{t-2}}}$, we can find a monochromatic combinatorial line in $\{1, 2\}^{N_{t-1}}$; let $z_{t-1} \in \{1, 2, *\}^{N_{t-1}}$ be the root of this line. As before, we have

$$c(y, z_{t-1}(1), z_t(a)) = c(y, z_{t-1}(2), z_t(b))$$

for all $y \in A^{S_{t-2}}$ provided $a = b$ or $1 \leq a, b \leq 2$. 
Proof of the Hales-Jewett Theorem

Continuing in this manner, we can take \( N_{t-u} \geq r^{(k-1)u} k^{s_{t-u}-1} \) and find \( z_{t-u} \in \{1, 2, *\}^{N_{t-u}} \) with

\[
c(y, z_{t-u}(a_0), z_{t-u+1}(a_1), \ldots, z_t(a_u)) = c(y, z_{t-u}(b_0), z_{t-u+1}(b_1), \ldots, z_t(b_u))
\]

for all \( y \in A^{s_{t-u}-1} \) provided \( a_i = b_i \) or \( 1 \leq a_i, b_i \leq 2 \) for all \( 0 \leq i \leq s \).
Proof of the Hales-Jewett Theorem

Once we have found $z_1, \ldots, z_t$, we can define an $r$-colouring of $\{2, \ldots, k\}^t$ by setting $c(a_1, \ldots, a_t) = c(z_1(a_1), \ldots, z_t(a_t))$. 

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By our inductive hypothesis, $H(k - 1, r)$ exists. Therefore, if $t \geq H(k - 1, r)$ then there is a monochromatic combinatorial line in $\{2, \ldots, k\}^t$; this corresponds to a monochromatic combinatorial line in $A^N$ since we can exchange any or all of the 2s in $(z_1(a_1), \ldots, z_t(a_t))$ for 1s without changing its colour.
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Thus, if $t \geq H(k-1, r)$ and each $N_i$ is defined as above, $N = N_1 + \ldots + N_t \geq H(k, r)$, which means that $H(k, r)$ exists.
Example

Suppose $t = 3$, $z_1 = 1^*$, $z_2 = 1^{**}2$, $z_3 = 1^*222$, and $z_3 = 11^{***}2$. If \{2452, 3453, 4454, \ldots\} is a monochromatic combinatorial line in \{2, \ldots, k\}^t$, then it induces the following monochromatic combinatorial line in $A^t$:

- $z(1) = 11\ 1442\ 15222\ 111112$
- $z(2) = 12\ 1442\ 15222\ 112222$
- $z(3) = 13\ 1442\ 15222\ 113332$
- $z(4) = 14\ 1442\ 15222\ 114442$
- $\vdots$
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Clearly, this theorem does not give very nice bounds. However, it provides the best known general bound for $H(k, r)$. 
Van der Waerden’s Theorem

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We label the least $n$ with this property $W(k, r)$. 
Proof of Van der Waerden’s Theorem

Let \( n = h(k - 1) + 1 \), where \( h = H(k, r) \), and fix an \( r \)-colouring \( c \) of \( \{1, \ldots, n\} \). Set \( A = \{0, 1, \ldots, k - 1\} \) and define \( f : A^h \to \{1, 2, \ldots, n\} \) to be the function
\[
f(x_1, \ldots x_h) = x_1 + x_2 + \ldots + x_h + 1.
\]
This function induces an \( r \)-colouring on \( A^h \), with \( c(x_1, \ldots, x_h) = c(f(x)) \).

Each combinatorial line \( L_\tau = \{\tau(0), \ldots, \tau(k - 1)\} \) in \( A^h \) rooted in \( \tau \) is mapped to an arithmetic progression of length \( k \) with \( a = f(\tau(0)) + 1 \) and \( d \) equal to the number of *s in \( \tau \). By the Hales-Jewett Theorem there is a monochromatic line in \( A^h \), which corresponds to a monochromatic arithmetic progression of length \( k \). Thus \( W(k, r) \leq h(k - 1) + 1 \).
Currently, the best known upper bound on $W(k, r)$ is

$$W(k, r) \leq 2^{2^22^k+9}$$

This bound follows from Timothy Gower’s 2001 proof of Szemerédi’s theorem (a stronger version of van der Waerden’s Theorem). Gower’s proof uses a combination of Fourier analysis and combinatorics.

The best known upper bound up to 2001 comes from the presented proof of the Hales-Jewett Theorem.