Tensor products and the Connes Embedding Problem

Jerrod M Smith

University of Regina
Department of Mathematics
Regina, Canada
smith36j@uregina.ca

June, 2011

Joint work with Dr. D.R. Farenick
Supported by NSERC USRA
Problems that have shaped mathematics:
Problems that have shaped mathematics:
  - Fermat’s Last Theorem [number theory]
Problems that have shaped mathematics:
- Fermat’s Last Theorem [number theory]
- Four Colour Theorem [graph theory]
Problems that have shaped mathematics:
- Fermat’s Last Theorem [number theory]
- Four Colour Theorem [graph theory]
- Poincaré Conjecture [geometric topology]
Problems that have shaped mathematics:

- Fermat’s Last Theorem [number theory]
- Four Colour Theorem [graph theory]
- Poincaré Conjecture [geometric topology]
- Riemann Hypothesis [EVERYTHING!!]
Problems that have shaped mathematics:

- Fermat’s Last Theorem [number theory]
- Four Colour Theorem [graph theory]
- Poincaré Conjecture [geometric topology]
- Riemann Hypothesis [EVERYTHING!!]
- Connes Embedding Problem (1976) [operator algebras]
**Definition**

A $C^*$-algebra is a normed, complete, involutive complex algebra $\mathcal{A}$ such that

1. $\|xy\| \leq \|x\|\|y\|
2. $\|x^* x\| = \|x\|^2$

for all $x, y \in \mathcal{A}$. 
\textbf{Definition}

A $C^*$-algebra is a normed, complete, involutive complex algebra $\mathcal{A}$ such that

1. $\|xy\| \leq \|x\|\|y\|

2. $\|x^*x\| = \|x\|^2$

for all $x, y \in \mathcal{A}$.

\textbf{Proposition}

$\|x^*\| = \|x\|$ \textit{for all} $x \in \mathcal{A}$ \textit{and if} $\mathcal{A}$ \textit{is unital then} $\|1\| = 1$. 
Examples of Hilbert space

1. $\mathbb{C}^n$, with inner product

$$\langle \xi, \eta \rangle = \sum_{j=1}^{n} \xi_j \bar{\eta}_j,$$

and norm induced by the inner product $||\xi|| = \langle \xi, \xi \rangle^{1/2}$
Examples of Hilbert space

1. $\mathbb{C}^n$, with inner product

$$\langle \xi, \eta \rangle = \sum_{j=1}^{n} \xi_j \bar{\eta}_j,$$

and norm induced by the inner product $||\xi|| = \langle \xi, \xi \rangle^{1/2}$

2. $L^2([a, b])$, with inner product

$$\langle f, g \rangle = \int_{a}^{b} f(t) \overline{g(t)} dt,$$

and norm

$$||f|| = \left( \int_{a}^{b} |f(t)|^2 dt \right)^{1/2}$$
Examples of $C^*$-algebras

1. $B(\mathcal{H}) = \{\text{continuous linear maps } x : \mathcal{H} \to \mathcal{H}\}$, involution $x \mapsto x^*$ defined by

$$\langle x\xi, \eta \rangle = \langle \xi, x^* \eta \rangle \ \forall \xi, \eta \in \mathcal{H}.$$ 

For all $x \in B(\mathcal{H})$ $\exists M > 0$ such that $\|x\xi\| \leq M\|\xi\|$ $\forall \xi$ the infimum of all such $M$ is the norm of $x$, denoted $\|x\|$. 

$\mathbb{C}$
Examples of $C^*$-algebras

1. $B(\mathcal{H}) = \{\text{continuous linear maps } x : \mathcal{H} \to \mathcal{H}\}$, involution $x \mapsto x^*$ defined by

$$\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle \ \forall \xi, \eta \in \mathcal{H}.$$ 

For all $x \in B(\mathcal{H})$ $\exists M > 0$ such that $||x\xi|| \leq M||\xi|| \ \forall \xi$ the infimum of all such $M$ is the norm of $x$, denoted $||x||$.

2. If $\dim \mathcal{H} = n$, i.e. $\mathcal{H} \cong \mathbb{C}^n$, then $B(\mathcal{H}) \cong M_n(\mathbb{C})$. 
Examples of $C^*$-algebras

1. $B(\mathcal{H}) = \{\text{continuous linear maps } x : \mathcal{H} \to \mathcal{H}\}$, involution $x \mapsto x^*$ defined by

$$\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle \quad \forall \xi, \eta \in \mathcal{H}.$$ 

For all $x \in B(\mathcal{H}) \exists M > 0$ such that $\|x\xi\| \leq M\|\xi\| \quad \forall \xi$ the infimum of all such $M$ is the norm of $x$, denoted $\|x\|$.

2. If $\dim \mathcal{H} = n$, i.e. $\mathcal{H} \cong \mathbb{C}^n$, then $B(\mathcal{H}) \cong M_n(\mathbb{C})$.

3. $C([a, b]) = \{f : [a, b] \to \mathbb{C} \mid f \text{ continuous}\}$ with involution

$$f^*(t) = \overline{f(t)}$$

and norm

$$\|f\| = \max_{t \in [a, b]} |f(t)|$$
A representation of a $C^*$-algebra $A$ on a Hilbert space $H$ is a function $\pi : A \rightarrow B(H)$ such that

1. $\pi$ is linear
2. $\pi(xy) = \pi(x)\pi(y)$
3. $\pi(x^*) = \pi(x)^*$
Representations of $C^*$-algebras

**Definition**

A representation of a $C^*$-algebra $A$ on a Hilbert space $\mathcal{H}$ is a function $\pi : A \rightarrow B(\mathcal{H})$ such that

1. $\pi$ is linear
2. $\pi(xy) = \pi(x)\pi(y)$
3. $\pi(x)^* = \pi(x^*)$

We can represent $C([a, b])$ on $L^2([a, b])$ as follows. Define a map by $\pi(f) = mf$, where $mf(g) = f \cdot g \ \forall g \in L^2([a, b])$.

Notice that

$$\|mf(g)\|^2 = \int_a^b |f(t)|^2 |g(t)|^2 \, dt \leq \left( \max_{t \in [a, b]} |f(t)| \right)^2 \int_a^b |g(t)|^2 \, dt$$

$$= \|f\|^2 \|g\|^2$$

Therefore $\|mf\| \leq \|f\|$, in fact $\|mf\| = \|f\|$.
Representations of $C^*$-algebras

Theorem (Gelfand-Naimark)

If $A$ is a $C^*$-algebra then $\exists$ a faithful representation $\pi : A \rightarrow B(\mathcal{H}_\pi)$ for some Hilbert space $\mathcal{H}_\pi$.

In other words, every $C^*$-algebra is a $C^*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. 

J.M. Smith

Tensor products and the Connes Embedding Problem
Representations of $C^*$-algebras

**Theorem (Gelfand-Naimark)**

If $A$ is a $C^*$-algebra then $\exists$ a faithful representation $\pi : A \rightarrow B(\mathcal{H}_\pi)$ for some Hilbert space $\mathcal{H}_\pi$.

In other words, every $C^*$-algebra is a $C^*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

**Remark**

The Gelfand-Naimark theorem is analogous to Cayley’s Theorem for groups, i.e., every group $G$ is a subgroup of a symmetric group.
Let $\mathbb{F}_\infty$ be the free group on countably many generators
$\{u_1, u_2, \ldots \}$; that is all finite words in $u_j, j \in \mathbb{N}$ and their
inverses.
Let $F_\infty$ be the free group on countably many generators \( \{u_1, u_2, \ldots \} \); that is all finite words in $u_j, j \in \mathbb{N}$ and their inverses.

Denote by $\mathbb{C}[F_\infty]$ the complex group algebra of $F_\infty$. 
Group \( \mathbb{C}^\ast \)-algebras

- Let \( \mathbb{F}_\infty \) be the free group on countably many generators \( \{u_1, u_2, \ldots\} \); that is all finite words in \( u_j, j \in \mathbb{N} \) and their inverses.
- Denote by \( \mathbb{C}[\mathbb{F}_\infty] \) the complex group algebra of \( \mathbb{F}_\infty \).
- \( \mathbb{C}[\mathbb{F}_\infty] \) is the free vector space on \( \mathbb{F}_\infty \) with multiplication induced by the multiplication on \( \mathbb{F}_\infty \).
Group $C^*$-algebras

- Let $F_\infty$ be the free group on countably many generators $\{u_1, u_2, \ldots\}$; that is all finite words in $u_j, j \in \mathbb{N}$ and their inverses.

- Denote by $\mathbb{C}[F_\infty]$ the complex group algebra of $F_\infty$.

- $\mathbb{C}[F_\infty]$ is the free vector space on $F_\infty$ with multiplication induced by the multiplication on $F_\infty$.

- Define involution $\ast$ on $\mathbb{C}[F_\infty]$ via $g^\ast = g^{-1} \ \forall g \in F_\infty$ and extending by linearity.
Let $\mathbb{F}_\infty$ be the free group on countably many generators \{${u_1, u_2, ...}$\}; that is all finite words in $u_j, j \in \mathbb{N}$ and their inverses.

Denote by $\mathbb{C}[\mathbb{F}_\infty]$ the complex group algebra of $\mathbb{F}_\infty$.

$\mathbb{C}[\mathbb{F}_\infty]$ is the free vector space on $\mathbb{F}_\infty$ with multiplication induced by the multiplication on $\mathbb{F}_\infty$.

Define involution $\ast$ on $\mathbb{C}[\mathbb{F}_\infty]$ via $g^* = g^{-1} \ \forall g \in \mathbb{F}_\infty$ and extending by linearity.

Any representation $\sigma : \mathbb{F}_\infty \to U(\mathcal{H}_\sigma)$ extends to an algebra homomorphism $\tilde{\sigma} : \mathbb{C}[\mathbb{F}_\infty] \to B(\mathcal{H}_\sigma)$.
Define a norm $\| \cdot \|$ on $\mathbb{C}[F_\infty]$ by

$$\|x\| = \sup\{||\tilde{\sigma}(x)|| \mid \sigma : F_\infty \to U(H_\sigma) \text{ representation}\}$$
Define a norm $\| \cdot \|$ on $\mathbb{C}[F_\infty]$ by

$$\| x \| = \sup \{ \| \tilde{\sigma}(x) \| \mid \sigma : F_\infty \to \mathcal{U}(\mathcal{H}_\sigma) \text{ representation} \}$$

Fact: $\| xy \| \leq \| x \| \| y \|$ and $\| x^* x \| = \| x \|^2$ for all $x, y \in \mathbb{C}[F_\infty]$. 

Complete $\mathbb{C}[F_\infty]$ in this norm to get a $\mathbb{C}^*$-algebra denoted $\mathbb{C}^*(F_\infty)$, called the full group $\mathbb{C}^*$-algebra of $F_\infty$.

Remark (Universal Property of $\mathbb{C}^*(F_\infty)$) For any countable collection of unitaries $\{v_j\} \subset B(\mathcal{H})$ there exists a *-homomorphism $\phi : \mathbb{C}^*(F_\infty) \to \mathbb{C}^*(\{v_j\})$ such that $\phi(u_j) = v_j$ for all $j \in \mathbb{N}$. 

J.M. Smith
Tensor products and the Connes Embedding Problem
Define a norm $\| \cdot \|$ on $\mathbb{C}[\mathbb{F}_\infty]$ by

$$\|x\| = \sup\{\|\tilde{\sigma}(x)\| \mid \sigma : \mathbb{F}_\infty \to \mathcal{U}(\mathcal{H}_\sigma) \text{ representation}\}$$

- Fact: $\|xy\| \leq \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$ for all $x, y \in \mathbb{C}[\mathbb{F}_\infty]$.

- Complete $\mathbb{C}[\mathbb{F}_\infty]$ in this norm to get a $C^*$-algebra denoted $C^*(\mathbb{F}_\infty)$, called the full group $C^*$ algebra of $\mathbb{F}_\infty$. 

J.M. Smith

Tensor products and the Connes Embedding Problem
Group $C^*$-algebras

- Define a norm $\| \cdot \|$ on $\mathbb{C}[F_\infty]$ by
  \[
  \|x\| = \sup \{ \|\tilde{\sigma}(x)\| \mid \sigma : F_\infty \to \mathcal{U}(\mathcal{H}_\sigma) \text{ representation} \}
  \]

- Fact: $\|xy\| \leq \|x\| \|y\|$ and $\|x^*x\| = \|x\|^2$ for all $x, y \in \mathbb{C}[F_\infty]$.

- Complete $\mathbb{C}[F_\infty]$ in this norm to get a $C^*$-algebra denoted $C^*(F_\infty)$, called the full group $C^*$ algebra of $F_\infty$.

**Remark (Universal Property of $C^*(F_\infty)$)**

For any countable collection of unitaries $\{v_j\} \subset B(\mathcal{H})$ there exists a $^*$-homomorphism $\varphi : C^*(F_\infty) \to C^*(\{v_j\})$ such that $\varphi(u_j) = v_j$ for all $j \in \mathbb{N}$. 
Definition

A $C^*$-algebra $A$ is called separable if there exists a countable dense subset of $U$ of $A$.

For example $\mathbb{C}$ is separable.
A $C^*$-algebra $\mathcal{A}$ is called separable if there exists a countable dense subset of $U$ of $\mathcal{A}$.

For example $\mathbb{C}$ is separable.

Corollary

For any separable $C^*$-algebra $\mathcal{A}$ there exists an ideal $J \subset C^*(F_\infty)$ such that $C^*(F_\infty)/J \cong \mathcal{A}$, i.e., $\exists$ a surjective homomorphism $\pi : C^*(F_\infty) \rightarrow \mathcal{A}$ such that $\mathcal{A} \cong C^*(F_\infty)/\ker \pi$

Every unital, separable $C^*$-algebra is a quotient of $C^*(F_\infty)$. 
Given associative \(*\)-algebras $A, B$ we can construct their tensor product as vector spaces as a quotient $A \otimes B = \mathbb{C}(A \times B)/\langle F \rangle_C$ where

$$F = \left\{ \begin{array}{l}
\alpha(a, b) - (\alpha a, b) \\
\alpha(a, b) - (a, \alpha b) \\
(a_1 + a_2, b) - (a_1, b) - (a_2, b) \\
(a, b_1 + b_2) - (a, b_1) - (a, b_2) \\
\end{array} \right\}$$
Given associative *-algebras $A, B$ we can construct their tensor product as vector spaces as a quotient $A \otimes B = \mathbb{C}(A \times B)/\langle F \rangle_{\mathbb{C}}$ where

$$F = \left\{ \begin{array}{l}
\alpha(a, b) - (\alpha a, b) \\
\alpha(a, b) - (a, \alpha b) \\
(a_1 + a_2, b) - (a_1, b) - (a_2, b) \\
(a, b_1 + b_2) - (a, b_1) - (a, b_2)
\end{array} \right\}$$

Denote by $a \otimes b = \{(a', b') \mid (a', b') - (a, b) \in \langle F \rangle_{\mathbb{C}}\}$, then

$$\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$$

$$\lambda(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

So far, this is just a complex vector space!
Tensor products of associative $\ast$-algebras

**Theorem**

If $B_A, B_B$ are bases for $A, B$ respectively then

$$\{ a \otimes b \mid a \in B_A, b \in B_B \}$$

is a basis for $A \otimes B$ as a vector space.
Theorem

If \( B_A, B_B \) are bases for \( A, B \) respectively then
\[
\{ a \otimes b \mid a \in B_A, b \in B_B \}
\]
is a basis for \( A \otimes B \) as a vector space.

We want to give \( A \otimes B \) the structure of a \(*\)-algebra. On \((A \otimes B)^2\)
there exists a bilinear function \( m \) such that
\[
m(a \otimes b, c \otimes d) = ac \otimes bd
\]

If \( \xi = \sum_{i=1}^{m} a_i \otimes b_i \), \( \eta = \sum_{j=1}^{n} c_j \otimes d_j \) then,
\[
\xi \eta = \sum_i \sum_j a_i c_j \otimes b_i d_j
\]

Define involution \((a \otimes b)^* = a^* \otimes b^*\)
Let $\mathcal{A}$, $\mathcal{B}$ be unital $C^*$-algebras. $\mathcal{A} \otimes \mathcal{B}$ is a complex, unital, associative, *-algebra.

**Definition**

A $C^*$-norm on $\mathcal{A} \otimes \mathcal{B}$ is a sub-multiplicative norm $\| \cdot \|_\alpha$ such that

$$\|x^*x\|_\alpha = \|x\|_\alpha^2$$

for all $x \in \mathcal{A} \otimes \mathcal{B}$.
Let $\mathcal{A}, \mathcal{B}$ be unital $C^*$-algebras. $\mathcal{A} \otimes \mathcal{B}$ is a complex, unital, associative, *-algebra.

**Definition**

A $C^*$-norm on $\mathcal{A} \otimes \mathcal{B}$ is a sub-multiplicative norm $\| \cdot \|_\alpha$ such that

$$\|x^*x\|_\alpha = \|x\|_\alpha^2 \text{ for all } x \in \mathcal{A} \otimes \mathcal{B}$$

**Definition**

By $\mathcal{A} \otimes_\alpha \mathcal{B}$ we mean the completion of $\mathcal{A} \otimes \mathcal{B}$ in the norm $\| \cdot \|_\alpha$

$\mathcal{A} \otimes_\alpha \mathcal{B}$ is a $C^*$-algebra
Max tensor product

**Definition (Max-norm)**

For $x \in \mathcal{A} \otimes \mathcal{B}$ we define

$$\|x\|_{\text{max}} = \sup_{\pi} \{ \|\pi(x)\|_{B(\mathcal{H}_\pi)} \mid \pi : \mathcal{A} \otimes \mathcal{B} \to B(\mathcal{H}_\pi) \}$$

Where the sup is taken over all unital $*$-homomorphisms $\pi$.
Max tensor product

**Definition (Max-norm)**

For \( x \in A \otimes B \) we define

\[
\|x\|_{\text{max}} = \sup_{\pi} \{ \|\pi(x)\|_{B(H_\pi)} \mid \pi : A \otimes B \to B(H_\pi) \}
\]

Where the sup is taken over all unital *-homomorphisms \( \pi \)

- \( \| \cdot \|_{\text{max}} \) is a C*-norm
- \( A \otimes_{\text{max}} B \) is called the maximal tensor product of \( A \) and \( B \)
Definition (Min-norm)

Fix faithful representations $\sigma : A \rightarrow B(\mathcal{H}_\sigma)$, $\rho : B \rightarrow B(\mathcal{H}_\rho)$ and define $\| \cdot \|_{\text{min}}$ on $A \otimes B$ by

$$\left\| \sum_{j=1}^{m} a_j \otimes b_j \right\|_{\text{min}} = \left\| \sum_{j=1}^{m} \sigma(a_j) \otimes \rho(b_j) \right\|_{B(\mathcal{H}_\sigma \otimes \mathcal{H}_\rho)}$$
Min tensor product

**Definition (Min-norm)**

Fix faithful representations $\sigma : \mathcal{A} \to B(\mathcal{H}_\sigma)$, $\rho : \mathcal{B} \to B(\mathcal{H}_\rho)$ and define $\| \cdot \|_{\text{min}}$ on $\mathcal{A} \otimes \mathcal{B}$ by

$$\| \sum_{j=1}^{m} a_j \otimes b_j \|_{\text{min}} = \| \sum_{j=1}^{m} \sigma(a_j) \otimes \rho(b_j) \|_{B(\mathcal{H}_\sigma \otimes \mathcal{H}_\rho)}$$

**Theorem**

The norm $\| \cdot \|_{\text{min}}$ is independent of the choice of $\sigma$ and $\rho$.

- $\| \cdot \|_{\text{min}}$ is a C*-norm
- $\mathcal{A} \otimes_{\text{min}} \mathcal{B}$ is called the minimal tensor product of $\mathcal{A}$ and $\mathcal{B}$
Some results

Theorem

\[ ||x||_{min} \leq ||x||_\alpha \leq ||x||_{max} \]

for all \( x \in A \otimes B \) and \( C^* \)-norms \( || \cdot ||_\alpha \).
Some results

**Theorem**

\[ \|x\|_{\text{min}} \leq \|x\|_\alpha \leq \|x\|_{\text{max}} \]

for all \( x \in A \otimes B \) and \( C^*\)-norms \( \| \cdot \|_\alpha \).

**Theorem**

There is a unique \( C^*\)-norm on \( A \otimes B \), for every \( C^*\)-algebra \( B \), if \( A \) is any one of the following \( C^*\)-algebras:

1. \( C([a, b]) \)
2. \( M_n(\mathbb{C}) \)

But not \( C^*(F_\infty) \) nor \( B(H) \) if \( \dim \mathcal{H} = \infty \).
Some more results

**Theorem (Kirchberg)**

There is a unique $C^*$-norm on $C^*(\mathbb{F}_\infty) \otimes B(\mathcal{H})$, i.e.,

$$C^*(\mathbb{F}_\infty) \otimes \min B(\mathcal{H}) = C^*(\mathbb{F}_\infty) \otimes \max B(\mathcal{H})$$
Some more results

Theorem (Kirchberg)

There is a unique $C^*$-norm on $C^*(\mathbb{F}_\infty) \otimes B(\mathcal{H})$, i.e.,

$$C^*(\mathbb{F}_\infty) \otimes \min B(\mathcal{H}) = C^*(\mathbb{F}_\infty) \otimes \max B(\mathcal{H})$$

Theorem (Junge-Pisier)

There exist at least two different $C^*$-norms on $B(\mathcal{H}) \otimes B(\mathcal{H})$, i.e.,

$$B(\mathcal{H}) \otimes \min B(\mathcal{H}) \neq B(\mathcal{H}) \otimes \max B(\mathcal{H})$$
The Kirchberg Problem (KP)

Question (KP)

Is there a unique C*-norm on $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$?

This open problem is logically equivalent to the Connes Embedding Problem (CEP).

We now describe a new matrix approach to solving KP and thus CEP.
Definition

An ordered *-vector space is a pair \((V, V_+)\) consisting of a *-vector space \(V\) and a subset \(V_+ \subset V_{sa}\) satisfying the two properties:

1. \(V_+\) is a cone in \(V_{sa}\), i.e., closed under \(+\) and scaling by \(\mathbb{R}_+\)
2. \(V_+ \cap (-V_+) = \{0\}\)

This gives an ordering on \(V\); we say that \(v < w\) if and only if \(w - v \in V_+\).
Definition

An ordered *-vector space is a pair \((V, V_+)\) consisting of a *-vector space \(V\) and a subset \(V_+ \subset V_{sa}\) satisfying the two properties:

1. \(V_+\) is a cone in \(V_{sa}\), i.e., closed under \(+\) and scaling by \(\mathbb{R}_+\)
2. \(V_+ \cap (-V_+) = \{0\}\)

This gives an ordering on \(V\); we say that \(v < w\) if and only if \(w - v \in V_+\).

Let \(\mathcal{C} = (C_n)_{n \in \mathbb{N}},\ C_n \subset M_n(V)\) for all \(n \in \mathbb{N}\) such that

1. \(C_n \oplus C_m \subset C_{n+m} \ \forall n, m \in \mathbb{N}\)
2. \(C_n \cap (-C_n) = \{0\} \ \forall n \in \mathbb{N}\)
3. \(\alpha^* C_n \alpha \subset C_p \ \forall n \times p \text{ complex matrix } \alpha, \ \forall n, p \in \mathbb{N}\)

Write \(M_n(V)_+ = C_n\) and call this set a positive matrix cone.

\((V, \mathcal{C})\) is called a matricially ordered space.
Matrix approach to KP ⇔ CEP

**Definition**

An order unit for \((V, C)\) is a distinguished element \(e \in V_+\) such that for all \(h \in V_{sa} = \{v \in V \mid v^* = v\}\) there exists \(r \in \mathbb{R}_+\) with

\[-re \leq h \leq re\]
Definition

An order unit for \((V, C)\) is a distinguished element \(e \in V_+\) such that for all \(h \in V_{sa} = \{v \in V \mid v^* = v\}\) \(\exists r \in \mathbb{R}_+\) with \(-re \leq h \leq re\)

Definition

We say that \(e\) is an Archimedean order unit if the only \(h \in V_{sa}\) such that \(h + \epsilon e \in V_+\) \(\forall \epsilon > 0\) is \(h \in V_+\)
Definition (Operator system)

An operator system is a matricially ordered space $(V, C)$ with an Archimedean order unit $e \in C_1$ such that

$$e_p = \begin{bmatrix} e & 0 \\ \vdots & \ddots \\ 0 & e \end{bmatrix}_{p \times p}$$

is an Archimedean order unit for $C_p \ \forall p \in \mathbb{N}$
Matrix approach to KP $\iff$ CEP

**Definition**

Let $\mathcal{E}_n = \{(a_{ij}) \in M_n(\mathbb{C}) \mid a_{ii} = a_{jj} 1 \leq i, j \leq n\}$ to the set of all complex matrices with constant diagonal. One can check that $\mathcal{E}_n$ is a concrete operator system.
Matrix approach to KP $\Leftrightarrow$ CEP

**Definition**

Let $\mathcal{E}_n = \{(a_{ij}) \in M_n(\mathbb{C}) \mid a_{ii} = a_{jj}, 1 \leq i, j \leq n\}$ to the set of all complex matrices with constant diagonal. One can check that $\mathcal{E}_n$ is a concrete operator system.

Farenick and Paulsen have shown that if $\mathcal{E}_n \otimes_{\text{min}} \mathcal{E}_n = \mathcal{E}_n \otimes_{\text{max}} \mathcal{E}_n$ for all $n \in \mathbb{N}$ then KP and CEP have affirmative solutions.
Matrix approach to KP ⇔ CEP

**Definition**

Let $\mathcal{E}_n = \{(a_{ij}) \in M_n(\mathbb{C}) \mid a_{ii} = a_{jj}, 1 \leq i, j \leq n\}$ to the set of all complex matrices with constant diagonal. One can check that $\mathcal{E}_n$ is a concrete operator system.

Farenick and Paulsen have shown that if $\mathcal{E}_n \otimes_{\min} \mathcal{E}_n = \mathcal{E}_n \otimes_{\max} \mathcal{E}_n$ for all $n \in \mathbb{N}$ then KP and CEP have affirmative solutions.

Currently, we’re working on the cases $n = 2, n = 3$. The goal is to determine whether or not the positive cones in the min-, max-norm coincide.
Example

It can be shown that \((E_2 \otimes \text{max} E_2)^+ \supset \text{conv}(E_2^+ \otimes E_2^+).\) Let \(\omega = e^{i\frac{2\pi}{3}}\) and let \(g\) be the matrix

\[
g = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
-1 & -1 & 1 & 1
\end{pmatrix} + \begin{pmatrix}
1 & -1 \\
-1 & 1 & 1 & 1
\end{pmatrix} + \omega \begin{pmatrix}
1 & -\omega & -\omega \\
-\omega & \omega & \omega & \omega
\end{pmatrix} + \omega^2 \begin{pmatrix}
1 & \omega & \omega \\
-\omega & -\omega & -\omega & -\omega
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 1 \\
-1 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1
\end{pmatrix}
\]
It can be shown that $(E_2 \otimes_{\text{max}} E_2)^+ \supset \text{conv}(E_2^+ \otimes E_2^+)$. Let $\omega = e^{i\frac{2\pi}{3}}$ and let $g$ be the matrix

$$g = \left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) + \left( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$+ \omega \left( \begin{bmatrix} 1 & -\omega^2 \\ -\omega^2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \omega^2 \\ \omega^2 & 1 \end{bmatrix} \right)$$

$$+ \omega^2 \left( \begin{bmatrix} 1 & \omega \\ \omega & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -\omega \\ -\omega & 1 \end{bmatrix} \right)$$
It can be shown that \((E_2 \otimes_{\max} E_2)^+ \supset \text{conv}(E_2^+ \otimes E_2^+)\). Let 
\(\omega = e^{i \frac{2\pi}{3}}\) and let \(g\) be the matrix

\[
g = \left( \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \otimes \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \right) + \left( \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \otimes \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \right) \\
+ \left( \omega \left( \left[ \begin{array}{cc} 1 & -\omega^2 \\ -\omega^2 & 1 \end{array} \right] \otimes \left[ \begin{array}{cc} 1 & \omega^2 \\ \omega^2 & 1 \end{array} \right] \right) \right) \\
+ \omega^2 \left( \left[ \begin{array}{cc} 1 & \omega \\ \omega & 1 \end{array} \right] \otimes \left[ \begin{array}{cc} 1 & -\omega \\ -\omega & 1 \end{array} \right] \right) \\
= \left[ \begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array} \right]
\]
Is KP/CEP logically equivalent to the matrix theoretic problem of determining if
\[ \mathcal{E}_n \otimes_{\min} \mathcal{E}_n = \mathcal{E}_n \otimes_{\max} \mathcal{E}_n \]
for all \( n \in \mathbb{N} \)?
Is KP/CEP logically equivalent to the matrix theoretic problem of determining if

$$\mathcal{E}_n \otimes_{\min} \mathcal{E}_n = \mathcal{E}_n \otimes_{\max} \mathcal{E}_n$$

for all $n \in \mathbb{N}$?

Thank you for your attention!