Bifurcations of an elastic ring with interacting particles

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Introduction

- To understand the wide array of problems from a variety of fields such that colloidal lithography, or to study the unusual fluid-like state of viruses, there are many models to study these problem and one of those is interacting particles moving on a manifold.
- In our problem, we study an elastic ring on which n particles are confined to move along. The problem is approached using control and optimization theory allowing us to simulate the behavior of the system. We study the possible bifurcations that occur for the two and three-particle cases when modifying the interaction strength and propose a general scheme of bifurcations that will happen for the general case.

Optimization problem

The system reaches equilibrium when its energy is minimized. In this case, the problem is to

$$\underset{x^i,u^i,l^i}{\text{minimize}} \sum_{\substack{i,j=1\\ j>i}}^{n} V(x^i(0),x^j(0)) + \sum_{i=1}^{n}\int_0^{l^i}\frac{1}{2}(u^i)^2\,ds^i$$

$$\text{subject to } \frac{dx^i}{ds^i} = \begin{bmatrix}\cos(x_3^i)\\ \sin(x_3^i)\\ u^i\end{bmatrix}$$

$$x^1(0) = x^n(l^n) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x^{i+1}(0) = x^i(l^i) \quad i \in \{1,\dots,n-1\}$$

$$\sum_{i=1}^{n} l^i = 1.$$

The position at arc length s^i on segment i will be denoted by $(x_1^i(s^i), x_2^i(s^i))$. The angle at arc length s^i between the horizontal axis and the tangent direction to the rod will be denoted by $x_3^i(s^i)$. Furthermore, $u^i(s^i)$ will denote the curvature of segment i at arc length s^i. The location of particle i is given by $x^i(0)$, and each pair of particles along the rod interact through the potential energy function

$$V(x^i,x^j) = V_0\left(\sqrt{(x_1^i-x_1^j)^2+(x_2^i-x_2^j)^2} - R_{eq}\right)^2.$$

Simulation for two and three particles

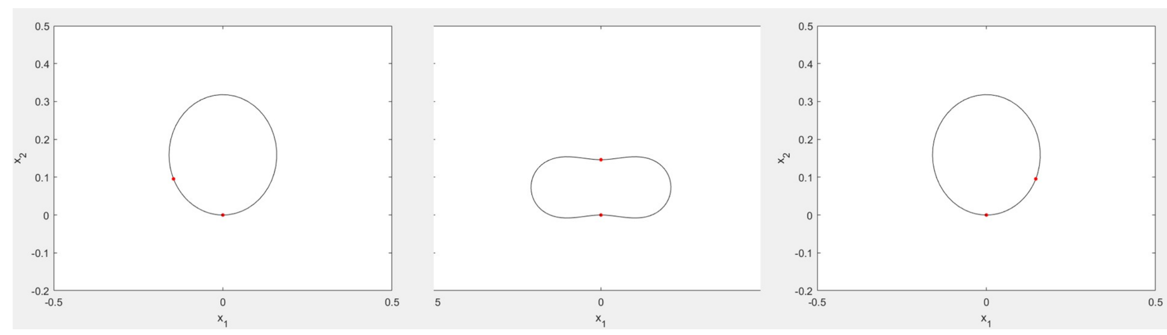

Figure 1: Three equilibria branches that arise from a pitchfork bifurcation.

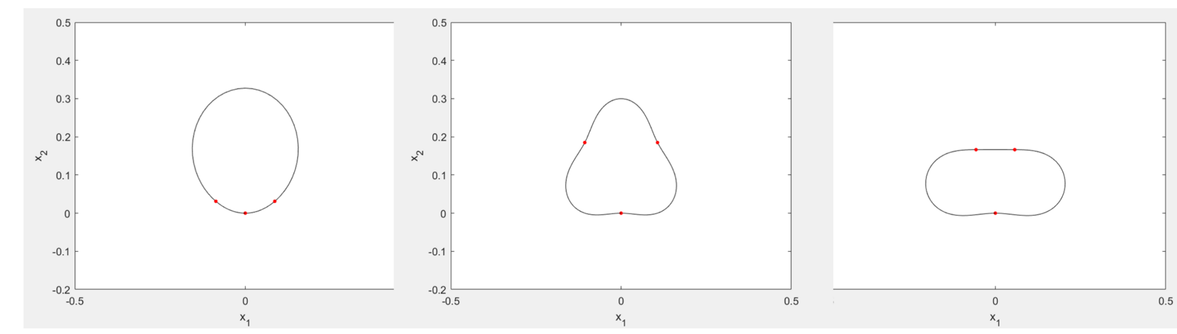

Figure 3: Three equilibria branches that arise from the first pitchfork bifurcation for the three-particle case.

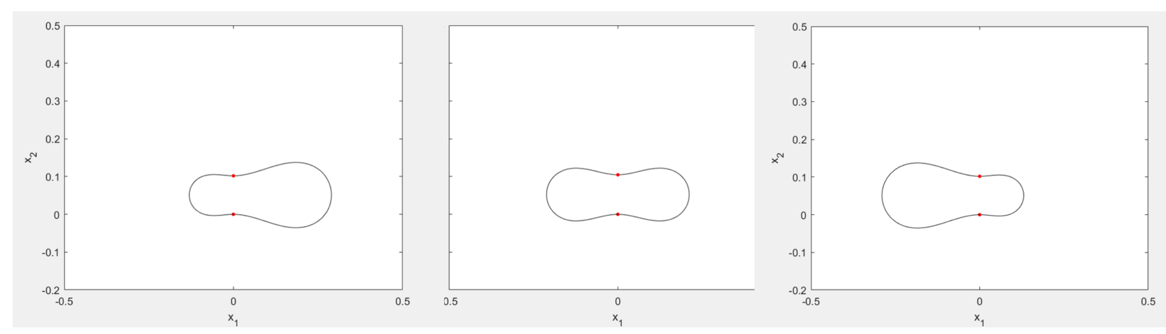

Figure 2: Three equilibria branches that arise from the second pitchfork bifurcation on 2 particles.

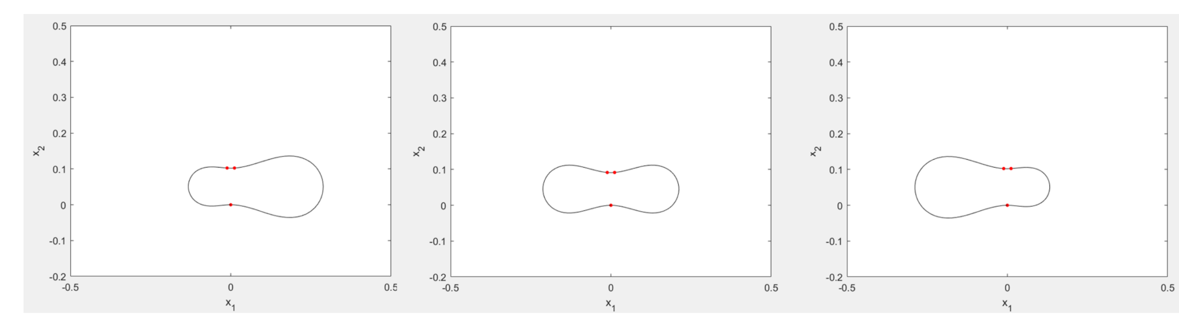

Figure 4: Three equilibria branches that arise from a pitchfork bifurcation on the right branch of Figure 3 .

Bifurcation diagrams for two and three particles

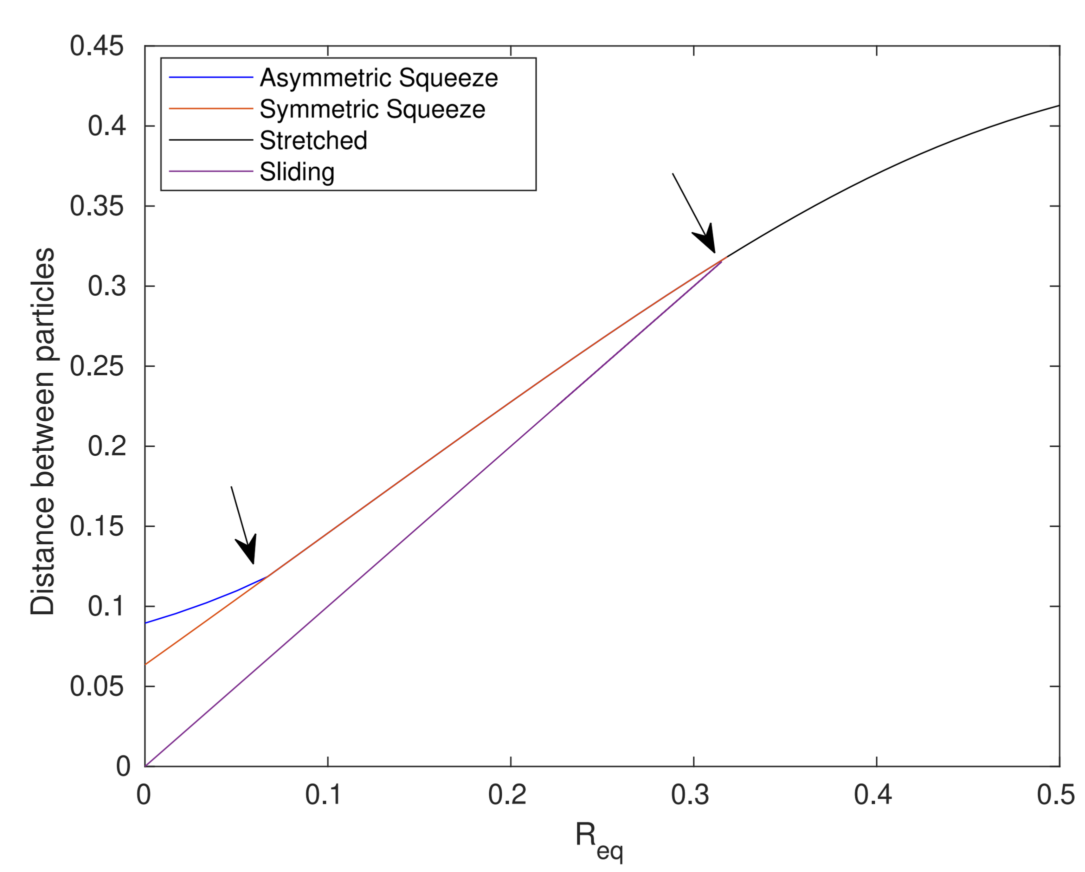

Figure 5: Bifurcation diagram for the two particles case with $V_0 = 2000$. There are two pitchfork bifurcations indicated by the two arrows. The left arrow is visualized on Figure 2. The right arrow is visualized on Figure 1.

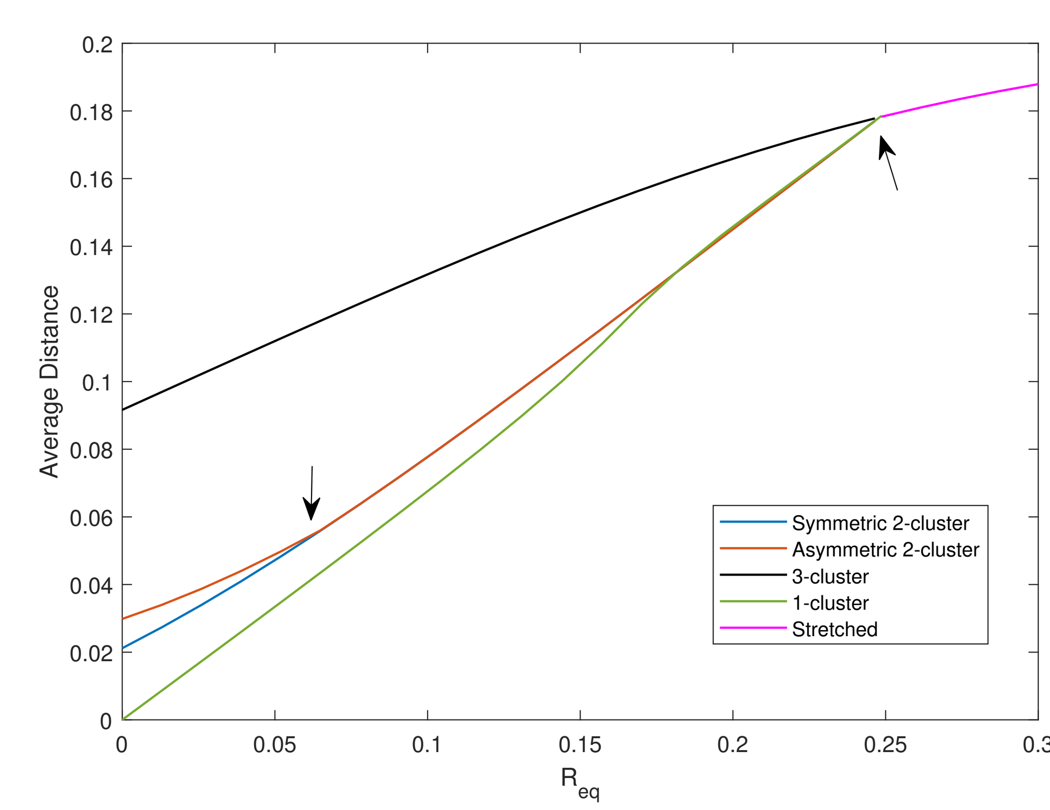

Figure 6: Bifurcation diagram for the three particles case with $V_0 = 1000$. There are two pitchfork bifurcations (modulo symmetries) indicated by the two arrows. The left arrow is visualized on Figure 4. The right arrow is visualized on Figure 3.

Results & Conclusions

- Propose a way of studying this minimization problem using control theory tools by reducing it to a boundary value problem of an ordinary differential equation.
- Characterize, for the two and three particle case, the bifurcation profiles for different interaction strength between the particles.
- Propose a general structure for the bifurcation profile in the general case of n particles, where different bifurcated branches correspond to different number of clusters of particles formed on the ring.

Future Plans

- Improve efficiency of the code, use dynamical system toolboxes (such as AUTO or CoCo), and/or run it on a computer cluster.
- Consider other potentials such as the Lennard-Jones Potential:

$$V_{LJ}(x^i,x^j) = \varepsilon\left[\left(\frac{r_{eq}}{r}\right)^{12} - \left(\frac{r_{eq}}{r}\right)^{6}\right].$$

- Analytically study the bifurcations happening at the circle solution.

References

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[2] S. H. Strogatz, Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering, Second edition. Boulder, CO: Westview Press, a member of the Perseus Books Group, 2015.

[3] T. Erber and G. M. Hockney, “Equilibrium configurations of N equal charges on a sphere,” J. Phys. A: Math. Gen., vol. 24, no. 23, pp. L1369–L1377, Dec. 1991, doi: 10.1088/0305-4470/24/23/008.

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