How to Turn a Needle in Small Areas

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Outline

1. Kakeya poses a problem

2. Besicovitch has a solution

3. More problems and solutions

§1
KAKEYA POSES A PROBLEM

The Needle Problem

In 1917, the Japanese mathematician Soichi Kakeya posed the following problem in 'Some Problems on Maximum and Minimum Regarding Ovals':

In the class of figures in which a segment of length 1 can be turned around through 360°, remaining always within the figure, which one has the smallest area?

Alternative phrasing: *. . . turned through 180° and returned to the original position with the ends reversed . . .*

Some Simple Figures to Start With

- **Circle** of diameter 1
  We can rotate the needle about its midpoint:

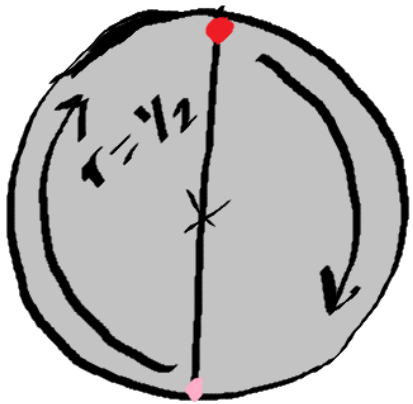

Figure 1: Fix the midpoint of the needle as the centre of the circle and spin.

In this case, the area is $\pi r^2 = \frac{\pi}{4} \approx 0.785$.

- **Reuleaux triangle** of diameter 1

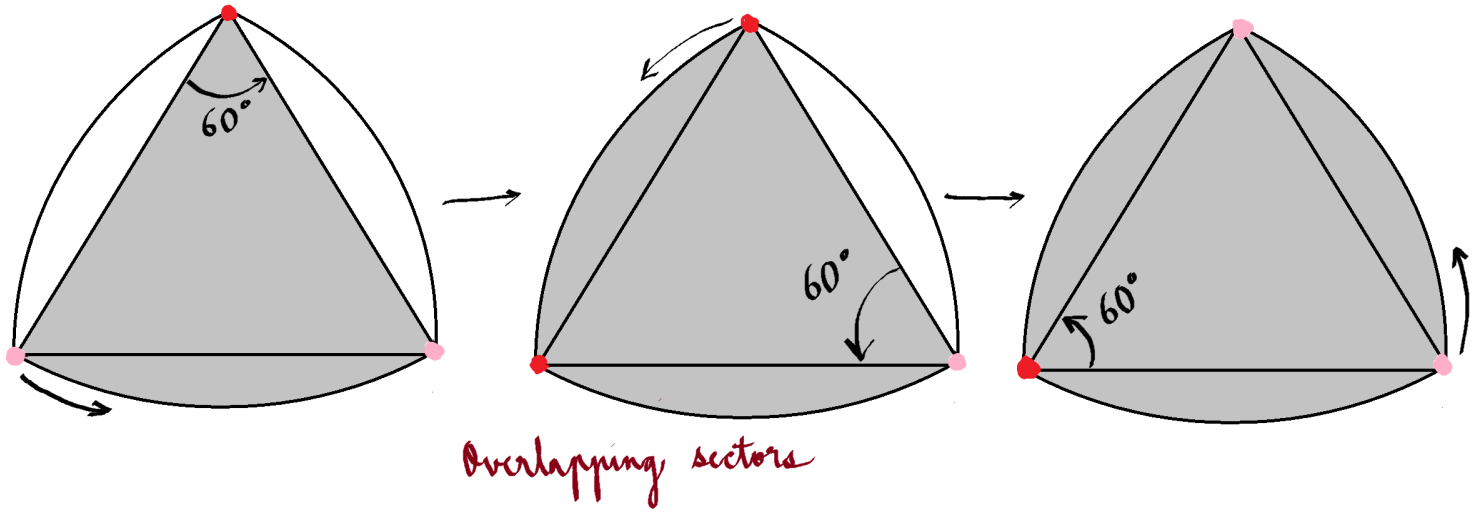

Figure 2: The boundary of the Reauleaux triangle is a curve of constant diameter.

This figure has area $3 \cdot \frac{\pi}{6} - 2 \cdot \frac{\sqrt{3}}{4} = \frac{\pi - \sqrt{3}}{2} \approx 0.705$.

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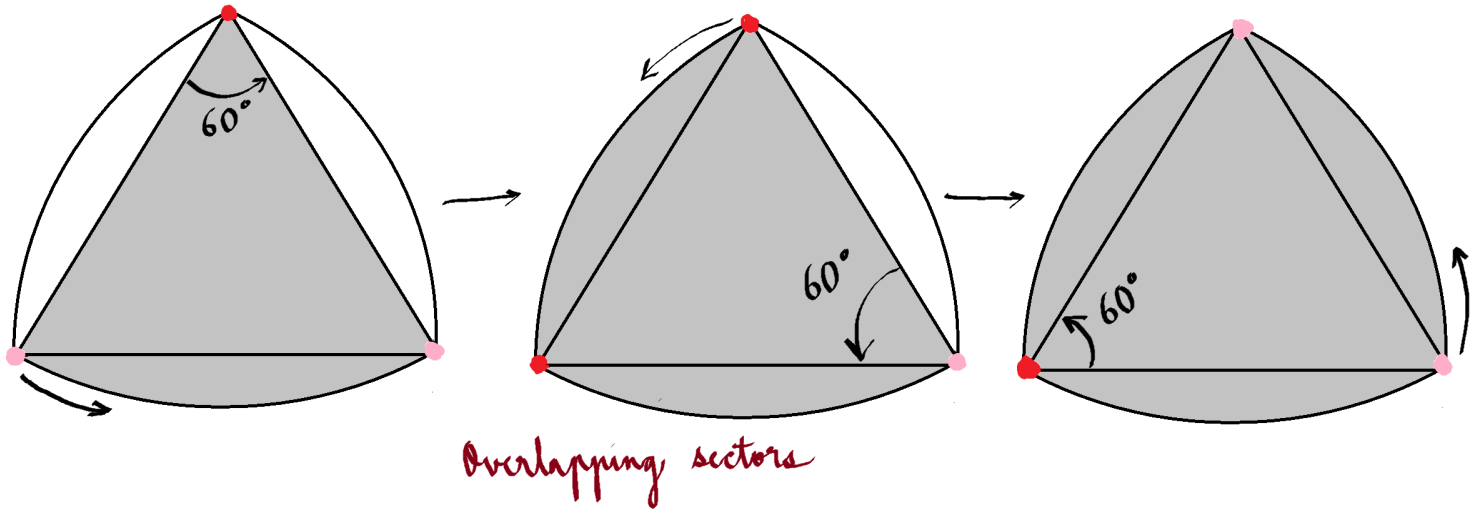

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But the motion of the needle is not limited to rotations! We can do better.

- **Equilateral triangle** of height 1

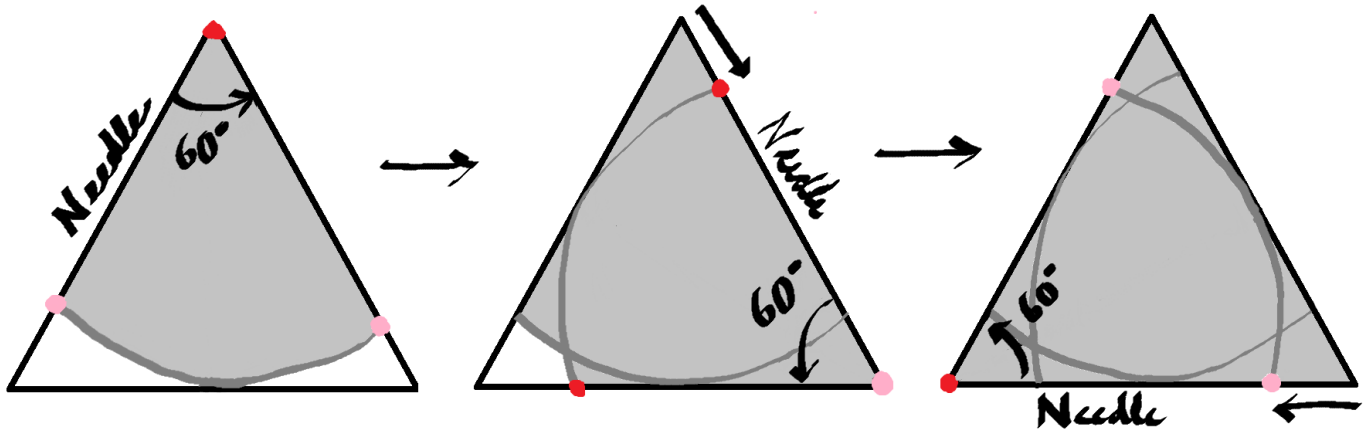

Figure 3: Use a combination of rotations and translations to sweep out an equilateral triangle.

One round around the triangle turns the needle through 180° and switches the ends when they are returned to the initial position.

In this case, the side length is $\frac{2}{\sqrt{3}}$, so the area is $\frac{1}{2}bh = \frac{1}{\sqrt{3}} \approx 0.577$.

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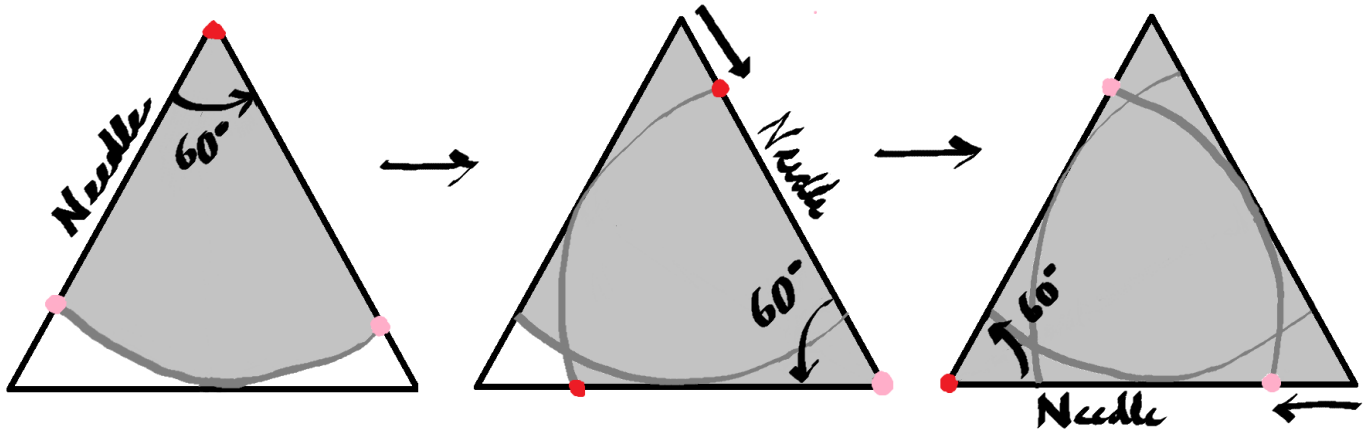

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In fact, this is the *smallest* figure when we impose a convexity condition (Pál, 1921).

Non-Convex Figures

- **Deltoid** inscribed in a circle of diameter $\frac{3}{2}$ by a smaller circle of radius $\frac{1}{2}$

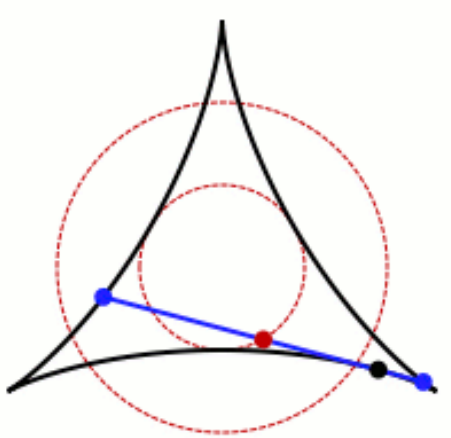

Figure 4: Deltoid — a hypocycloid with three cusps by Claudio Rocchini (Wikipedia)

Here, the area is $\frac{\pi}{8} \approx 0.393$.

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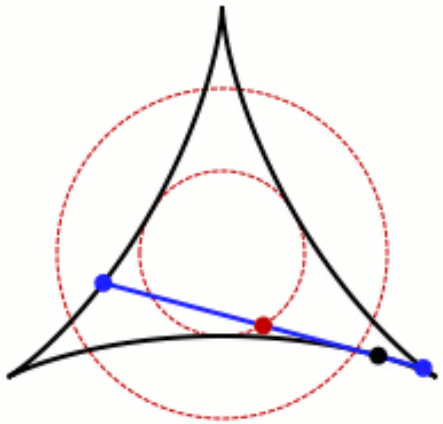

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Kakeya believed this was the *smallest* area required to turn a needle, but he did not have a proof.

Enter Besicovitch

Unbeknownst to S. Kakeya, there was a Russian mathematician named Abram Samoilovitch Besicovitch who, at the same time, was working on a different problem:

Given a function of two variables, Riemann-integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that the repeated simple integration along the two directions exists and gives the value of the integral over the domain?

He constructed a counter example and published his result in 1920.

‘Twin Problems’

The problem amounted to showing the existence of a compact set with measure zero containing line segments in every direction with lengths no less than 1.

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Answer to Kakeya’s Question

There exist figures with **arbitrarily small** area that allow a needle to continuously turn 360° while remaining within them.

§2
BESICOVITCH HAS A SOLUTION

Besicovitch Set

Definition

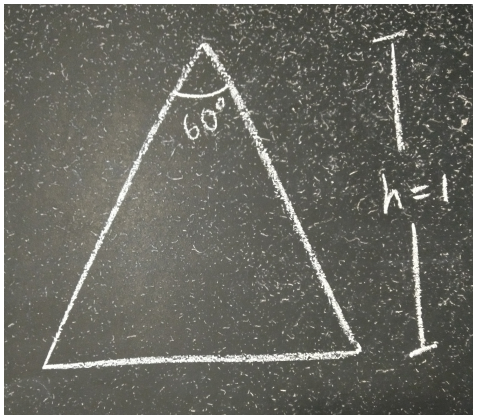
A **Besicovitch set** is a union of unit segments in every direction.

(Some definitions require measure zero and compactness.)
A **Kakeya needle set** is a Besicovitch set with an additional condition for continuity.

We're going to look at one such set: the Perron tree (more accurately, the Besicovitch–Perron–Schoenberg–. . . tree).

How to Draw a Perron Tree – a Sketch

- We start off by splitting up the problem into chunks where we can turn the needle through 60°.
- An equilateral triangle of height 1 works.



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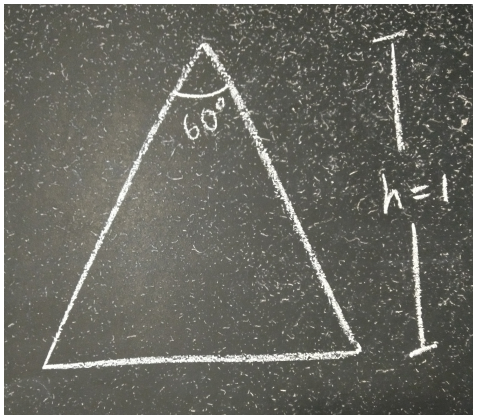

Figure 5: Step 1 in making a BPS tree

- Now let's forget about the need to turn the needle continuously and think about how to reduce the area for a bit.

Creating an Overlap

- What if we cut the triangle and overlap them?

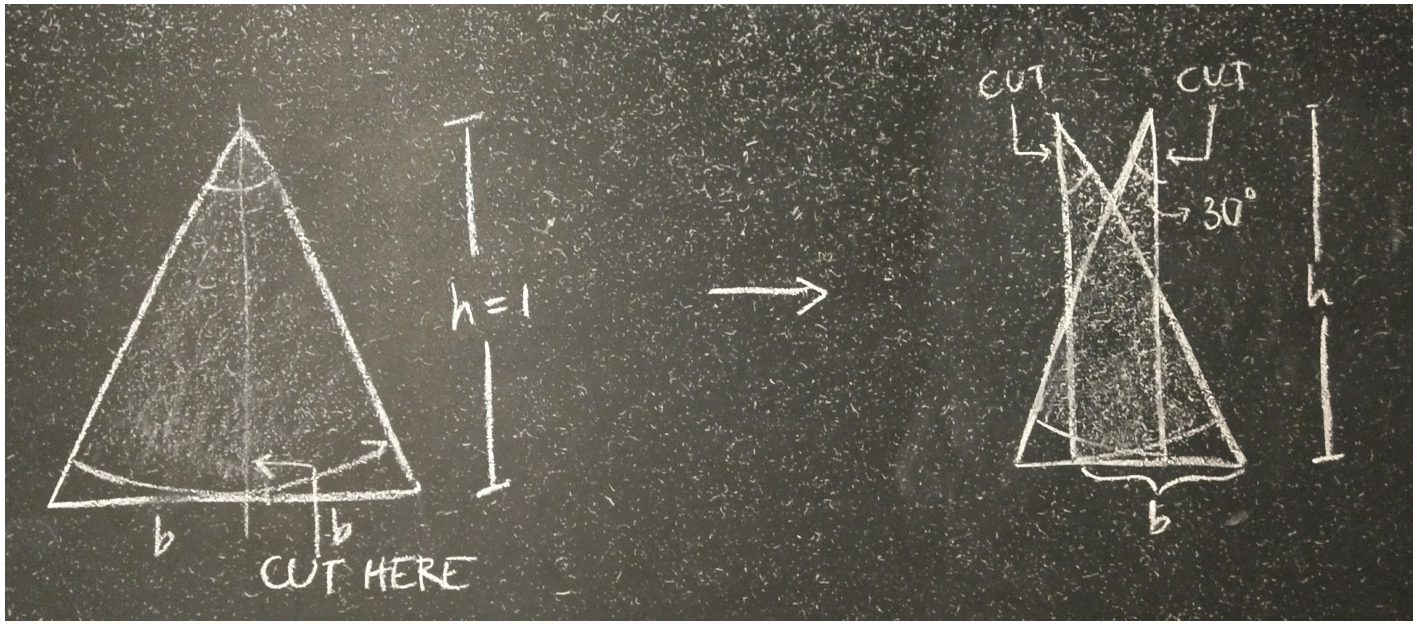

Figure 6: Step 2 in making a Besicovitch–Perron tree

- Halve the base of the triangle and cut along the median.
- We still sweep out 60° but with a discontinuity at the cut (ignore for now).

Area Reduced by Overlap

Lemma 1

Let $s \in \mathbb{R}$ be a parameter such that $\frac{1}{2} < s < 1$. Let T be a triangle with base length $2b$ and unit height h. It is divided into two subtriangles T_1 and T_2 along the median intersecting the base. Slide T_2 over T_1 along the base so that the length of the overlap in the base is $2b(1-s)$.

Then the new figure has area $(s^2 + 2(1-s)^2)A(T)$ where $A(T)$ is the area of T.

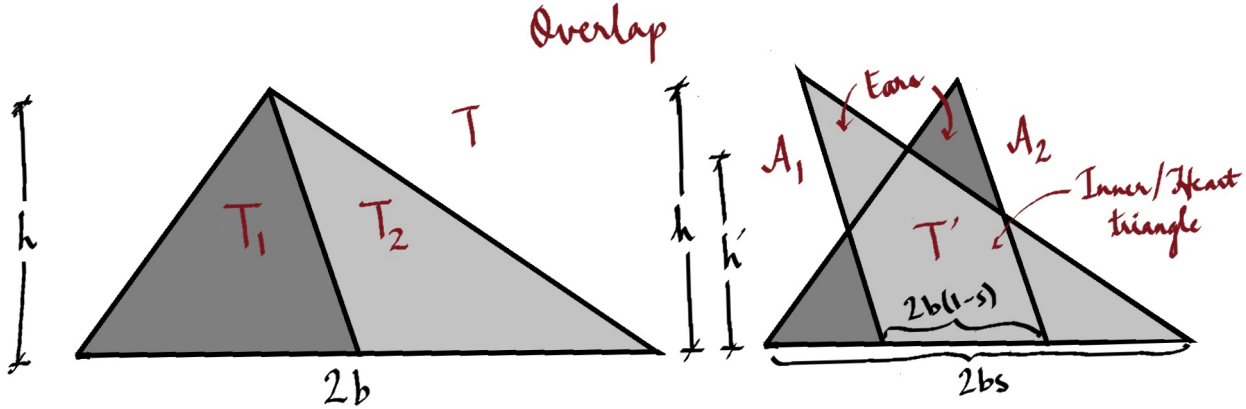

Figure 7: Reduction in area

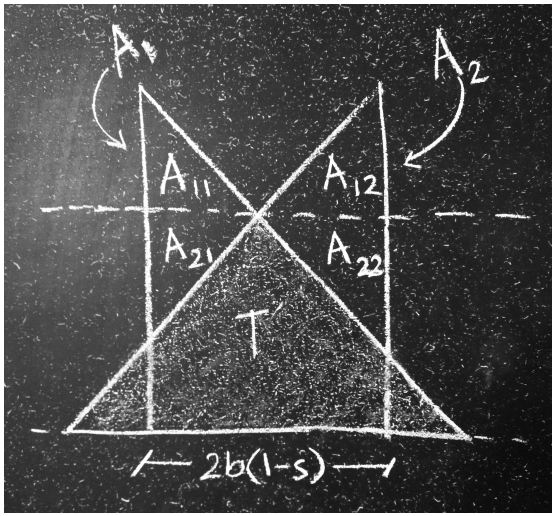

Figure 8: Line parallel to the base through the 'ears'

Sketch of the proof. The inner triangle T' is similar to T and has base length $2bs$. Then its area is $s^2A(T)$.
The smaller triangles above T' ('ears') can both be split into four by a line parallel to the base. Here, T_1 is similar to A_{12} and T_2 is similar to A_{21}, both with ratios $1-s$. In addition, $A_{11} \cong A_{22}$ and $A_{12} \cong A_{21}$. So each half-ear has area $\frac{1}{2}(1-s)^2A(T)$.
The total area of the new figure is $(s^2 + 2(1-s)^2)A(T)$ as required.

How Many Cuts to Make a Tree?

- We want pairs of triangles so we can overlap them. If we have a pair of overlaps, we can overlap them too.

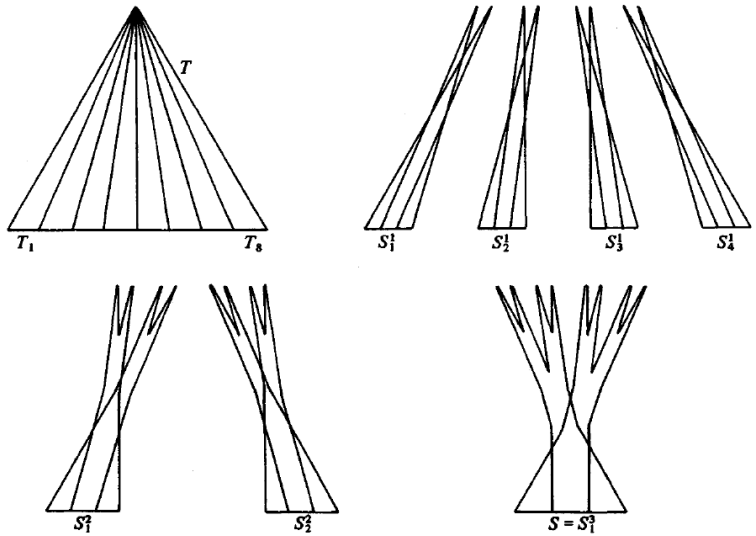

Figure 9: Falconer, *The Geometry of Fractal Sets* (1985).

- 2, 4, 8, 16, ...
- We want 2^n subtriangles with equal base lengths for some $n \in \mathbb{Z}$ to get nice overlaps.
- Choose n large enough to make iterated overlap area small.
- After n overlaps, get a tree.

How Small is the Tree?

- Fix the translation parameter $\frac{1}{2} < r < 1$.
- Make subtriangles $S_1^0, \ldots S_{2^k}^0$ with equal base lengths $2^{1-k}b$.
- Split into pairs S_{2i-1}^0, S_{2i}^0 with $1 \leq i \leq 2^{k-1}$ and overlap them like in the lemma.

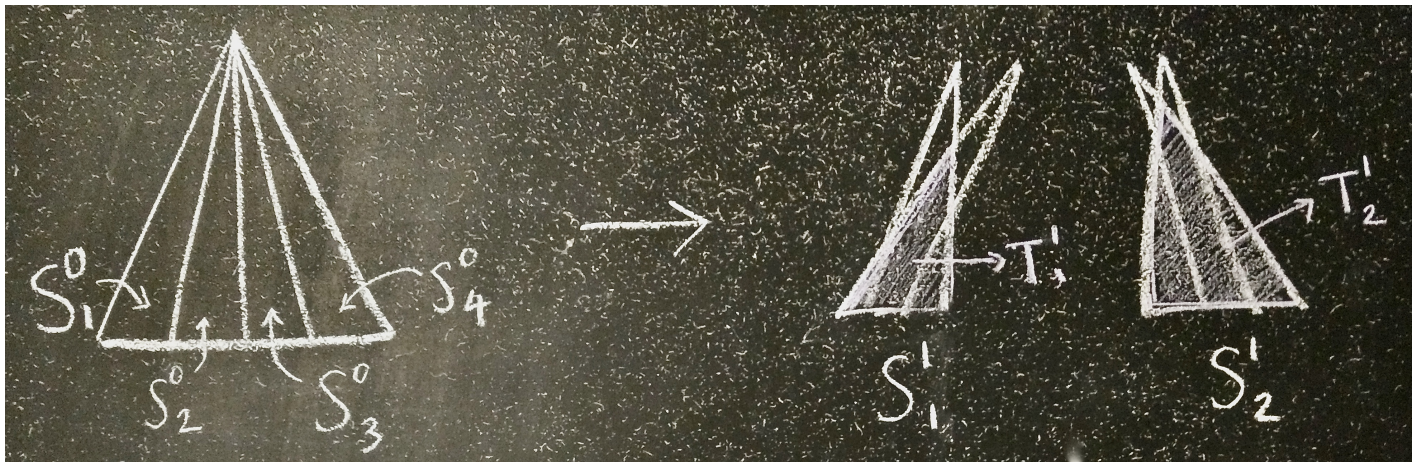

Figure 10: The first round of overlaps

- After the first round of overlaps, call the new figures S_i^1 where $1 \leq i \leq 2^{n-1}$ (because we've halved the number of figures). Recall that the inner triangle T_i^1 in each case is similar to the original triangle (here the union of a pair S_{2i-1}^0, S_{2i}^0).
From the lemma,

$$S_i^1 = (r^2 + 2(1-r)^2)A(S_{i-1}^0 \cup S_i^0)$$

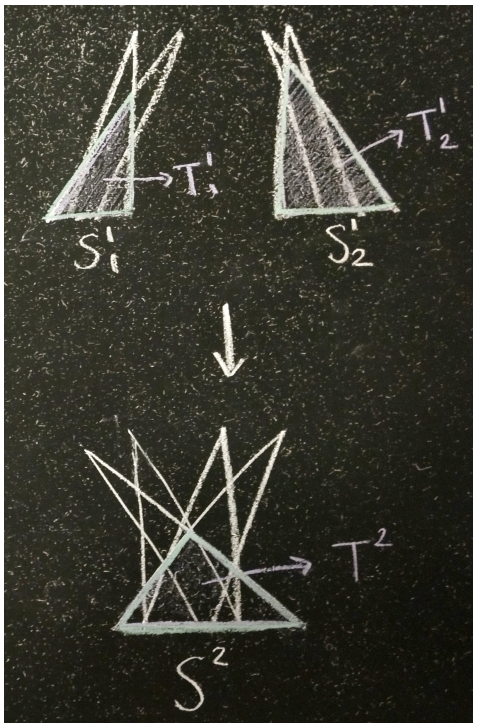

Figure 11: The second round of overlaps

- Now go through a second round of pairing and overlapping the S_j^1 triangles with $1 \leq j \leq 2^{k-2}$. Each new figure has four spikes where two are old and two are new and a new inner triangle T_j^1. There is some overlap between those parts, but we can get an upper bound by using the lemma and simple geometry:

$$A(T_j^1 \cup \text{new arms}) = r^2(r^2 + 2(1-r)^2 A(S_{4i-3}^0 \cup \cdots \cup S_{4i}^0))$$

$$\vdots$$

$$\sum_{j=1}^{2^{k-2}} A(S_j^2) \leq (r^4 + 2r^2(1-r)^2 + 2(1-r)^2)A(T)$$

- In round m, we have

$$\sum_{j=1}^{2^{k-m}} A(S_j^m) \leq \left(r^{2m} + 2(1-r)^2 \sum_{j=0}^{2m} r^{2m} \right) A(T)$$

- Finally, after k iterations, use geometric series to get

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But what about continuity?!

Repairing the Cuts for Smooth Turning

- Recall Besicovitch had to fix continuity to modify his result to solve this problem.
- The trick is **Pál joins.**

Final Assembly

Given $\varepsilon > 0$,

- Pick large enough k and r close enough to 1 such that

$$r^{2k} A(T) < \frac{\varepsilon}{12}$$
$$2(1-r)A(T) < \frac{\varepsilon}{12}$$

Then the final tree has area $< \frac{\varepsilon}{6}$.

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- Make each of the $2^k - 1$ Pál joins smaller than $\frac{\varepsilon}{6(2^k)}(2^k - 1)$.
- Then this whole figure has area less than $\frac{\varepsilon}{6} + \frac{\varepsilon}{6(2^k)}(2^k - 1) < \varepsilon/3$ This shows that we can make the area in which the 60° turn happens as small as you want.

- Glue 3 of these figures together to get a figure in which the needle can be turned 360° and return to the starting position with reversed orientation. Then

$$\text{Area of the glued trees} < 3 \cdot \frac{\varepsilon}{3} < \varepsilon$$

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- (Technically, you would need to glue 2 additional Pál joins between these 3 figures to have continuous turning before you are done.)

§3
MORE PROBLEMS AND SOLUTIONS

Making Smaller Sets

- Kakeya (needle) sets must have some small positive area but Besicovitch/Kakeya sets can have measure zero.

  Other ways of constructing a Besicovitch set: Kahane (1969) uses the Cantor set to make one with zero area.

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- Radius of a simply-connected Kakeya needle set:
  - There are arbitarily small Kakeya sets within a circle with radius $2 + \epsilon$ (van Alphen, 1941).
  - Given $\epsilon > 0$, there exists a simply connected Kakeya set with area less than ϵ within a circle of radius 1 (Cunningham, 1971).

What Next? Things You Might Find Interesting

- **Kakeya conjecture** Besicovitch sets in R^n must have Minkowski and Hausdorff dimension n.
  - Stronger formulation involving maximal functions
  - Proved for $n = 2$ (Davies, 1971) but an open problem for higher n.
  - Finite field Kakeya conjecture (Dvir, 2009). Let $E \in F^n$ be a Kakeya set. Then E has cardinality at least $c_n |F^n|$ where $c_n > 0$ depends only on n.
- **Ball multiplier conjecture (disk conjecture) counterexample** shows fourier transform does not converge in $L^p(R^n)$ spaces for $n > 1$ except at $p = 2$ (Fefferman, 1971).

Further Reading (and a film)

- Terry Tao's blog
- Izabella Laba's website
- T. Wolff, *Recent work connected with the Kakeya problem.* Prospects in Mathematics, AMS, (1999), pp. 129–162.
- F. Cunningham, *The Kakeya Problem for Simply Connected and for Star-Shaped Sets.* The American Mathematical Monthly 78 (1971), no. 2, pp. 114-29.
- The AMS film about the Kakeya problem featuring Besicovitch

The History of the Problem (if you can find/read some of these!)

- S. Kakeya, *Some problems on maximum and minimum regarding ovals*. Tohoku Science Reports 6 (1917), pp. 71–88.
- A. S. Besicovitch, *On Kakeya's problem and a similar one*. Math. Zeitschrift 27 (1928), 312–320.
- G. Pál, *Ein Minimumproblem für Ovale*. Math. Ann. 83 (1921), pp. 311–319.
- O. Perron, *Über einen Satz von Besicovitch*. Math. Zeitschrift 28 (1928), pp. 383–386.
- If you can read German, good for you.

References

- K. J. Falconer, *The Geometry of Fractal Sets*. Cambridge University Press, 1985.
- M. de Guzman, *Differentiation of Integrals in R^n*. Springer, 1975.
- S. G. Krantz, *A Panorama of Harmonic Analysis*. The Mathematical Association of America, 1999.
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- I. J. Schoenberg, *On the Besicovitch–Perron solution of the Kakeya problem.* In: *Studies in Mathematical Analysis and Related Topics: Essays in Honor of George Polya*, pp. 359–363. Stanford University Press, 1962.

Thank you!

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